

# Some geometric structures of wave equations on manifolds of ‘neutral signatures’

Jean-Juste Bashingwa and A. H. Kara<sup>1</sup>

School of Mathematics, University of the Witwatersrand, Johannesburg, South Africa.

## Abstract

In this paper, we study the symmetries and perform other geometric analyses of the wave equation on some spacetimes with non diagonal metric  $g_{ij}$  which are of neutral signatures. Wave equations on the standard Lorentzian manifolds have been done but not on the manifolds from metrics of neutral signatures.

*Keywords:* Manifolds; neutral signatures; wave equations

## 1 Introduction

The symmetry classification problem for a number of wave equations has been studied in flat space [17, 18, 2, 9] and non flat space (non-zero constant curvature) [1, 8]. In this work we pursue an investigation of symmetries of the wave equation on some spacetimes **with non diagonal metric  $g_{ij}$  and neutral signatures**. The corresponding ASD (anti self dual) manifolds have been studied in detail by a number of authors, for e.g., Fukaya [5], Dunajski [4], Plebanski [14] and Malykh et al [12], inter alia.

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<sup>1</sup>Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

One of the more significant applications of Lie symmetry groups is to achieve a complete classification of symmetry reductions of partial differential equations. The standard wave equation in  $(3 + 1)$ -dimensions has been extensively studied in the literature. A detailed symmetry analysis of this equation is discussed in [7, 16].

## 2 Wave equations on the ASD-Einstein manifold

A detailed symmetry analysis of ASD-Einstein manifolds has been done in [3]. The metric on the ASD Ricci-flat is locally given by

$$ds^2 = dzdy + dt dx - \left( -\frac{3z}{t^2}y + \frac{x}{t} \right) dt^2 + \frac{2y}{t} dt dz$$

A Gordon type equation in a curved space or in curvilinear coordinates is given by the expression

$$\square u = \frac{1}{\sqrt{|g|}} \left( g^{ab} u_{,b} \sqrt{|g|} \right)_a = k(u) \quad (2.1)$$

$g_{ab}$  being the metric of this space,  $g = ||g_{ab}||$  its determinant,  $g^{ab}$  the inverse of  $g_{ab}$ , where  $_{,b} = \partial_b$  and  $\square$  is the D'Alembertian, sometimes called the "box" operator.

Consequently, the Gordon type equation on the ASD manifold takes the form

$$\frac{xt - 3yz}{t^2} u_{xx} - \frac{2y}{t} u_{xy} + u_{x,t} + u_{yz} - k(u) = 0. \quad (2.2)$$

### 2.1 Lie symmetries

The procedure for finding Lie point symmetries is well known [15] and thus will be presented without details. It turns out, from the symmetry study, that some special

polynomial cases of  $k(u)$  arise. Consequently, we study four cases for  $k(u)$  in (2.2) given by

(i)  $k(u) = 0$  (wave equation)

(ii)  $k(u) = u$

(iii)  $k(u) = u^3$

(iv)  $k(u) = u^n, \quad n \neq 0, 1, 3$

**Case (i)**  $k(u) = 0$ . Following the symmetry criterion, we find that equation (2.2) in this case admits the following fifteen Lie point symmetries,

$$\begin{aligned}
X_1 &= u\partial_u \\
X_2 &= f_1(x, y, z, t)\partial_u \\
X_3 &= t\partial_x \\
X_4 &= 3t^{-1-\sqrt{7}}z\partial_x + (2 + \sqrt{7})t^{-\sqrt{7}}\partial_y \\
X_5 &= 3t^{-1+\sqrt{7}}z\partial_x - (-2 + \sqrt{7})t^{\sqrt{7}}\partial_y \\
X_6 &= t\partial_t + y\partial_y \\
X_7 &= 2y\partial_y + 3t\partial_t - x\partial_x \\
X_8 &= (-2 + \sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_9 &= (2 + \sqrt{7})t^{1+\sqrt{7}}y\partial_x - t^{2+\sqrt{7}}\partial_z \\
X_{10} &= t^{-\sqrt{7}} \left[ (1 - \sqrt{7})uy\partial_u + \sqrt{7}y^2\partial_y + tx\partial_z - ty\partial_t + \frac{(-3+\sqrt{7})txy+3y^2z}{t}\partial_x \right] \\
X_{11} &= t^{\sqrt{7}} \left[ (-1 - \sqrt{7})uy\partial_u + \sqrt{7}y^2\partial_y - tx\partial_z + ty\partial_t + \frac{(3+\sqrt{7})txy-3y^2z}{t}\partial_x \right] \\
X_{12} &= 2t\partial_t + y\partial_y + z\partial_z \\
X_{13} &= \frac{t^2x^2-txyz+3y^2z^2}{t^3}\partial_x - \frac{u(tx+yz)}{t^2}\partial_u + \frac{xz}{t}\partial_z + \frac{y(tx+2yz)}{t^2}\partial_y - \frac{yz}{t}\partial_t \\
X_{14} &= t^{-2-\sqrt{7}} \left[ (1 - \sqrt{7})uz\partial_u + (-2 + \sqrt{7})z^2\partial_z - tz\partial_t + (tx + 2yz)\partial_y + \frac{(-3+\sqrt{7})txz+3yz^2}{t}\partial_x \right] \\
X_{15} &= t^{-2+\sqrt{7}} \left[ (-1 - \sqrt{7})uz\partial_u + (2 + \sqrt{7})z^2\partial_z + tz\partial_t - (tx + 2yz)\partial_y + \frac{(3+\sqrt{7})txz-3yz^2}{t}\partial_x \right]
\end{aligned} \tag{2.3}$$

where

$$t^2 f_{1,yz} + t^2 f_{1,xt} - 2tyf_{1,xy} + tx f_{1,xx} - 3yz f_{1,xx} = 0$$

**Case (ii)**  $k(u) = u$ . Equation (2.2) in this case admits eight Lie point symmetries given by,

$$\begin{aligned} X_1 &= u \partial_u \\ X_2 &= f_1(x, y, z, t) \partial_u \\ X_3 &= t \partial_x \\ X_4 &= -\frac{z(2f_2[t] + t f_2')}{t} + f_2[t] \partial_y \\ X_5 &= (-2 + \sqrt{7}) t^{1-\sqrt{7}} y \partial_x + t^{2-\sqrt{7}} \partial_z \\ X_6 &= -(2 + \sqrt{7}) t^{1+\sqrt{7}} y \partial_x + t^{2+\sqrt{7}} \partial_z \\ X_7 &= y \partial_y - z \partial_z \\ X_8 &= t \partial_t - x \partial_x \end{aligned} \tag{2.4}$$

where

$$-t^2 f_1 + 2t^2 f_{1,yz} + 2t^2 f_{1,xt} - 4tyf_{1,xy} + 2tx f_{1,xx} - 6yz f_{1,xx} = 0$$

and

$$7f_2 - t f_2' - t^2 f_2'' = 0$$

**Case (iii)**  $k(u) = u^3$ . Equation (2.2) in this case admits fourteen Lie point symmetries given by,

$$\begin{aligned}
X_1 &= t\partial_x \\
X_2 &= -f_1(t)\partial_y + z\left(\frac{2f_1(t)}{t} + f_1'\right)\partial_x \\
X_3 &= (-2 + \sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_4 &= (2 + \sqrt{7})t^{1+\sqrt{7}}y\partial_x - t^{2+\sqrt{7}}\partial_z \\
X_5 &= -(-1 + \sqrt{7})t^{-\sqrt{7}}uy\partial_u + \sqrt{7}t^{-\sqrt{7}}y^2\partial_y + t^{1-\sqrt{7}}x\partial_z - t^{1-\sqrt{7}}y\partial_t + t^{-1-\sqrt{7}}y((-3 + \sqrt{7})xt + 3yz)\partial_x \\
X_6 &= -(1 + \sqrt{7})t^{\sqrt{7}}uy\partial_u + \sqrt{7}t^{\sqrt{7}}y^2\partial_y - t^{1+\sqrt{7}}x\partial_z + t^{1+\sqrt{7}}y\partial_t + t^{-1+\sqrt{7}}y((3 + \sqrt{7})xt - 3yz)\partial_x \\
X_7 &= y\partial_y - z\partial_z \\
X_8 &= 2z\partial_z + t\partial_t - u\partial_u + x\partial_x \\
X_9 &= 2x\partial_x + 2z\partial_z - u\partial_u \\
X_{10} &= t^{-2\sqrt{7}}\left[(-1 + \sqrt{7})u\partial_u - (-2 + \sqrt{7})z\partial_z - \sqrt{7}y\partial_y + \frac{tx+2(-7+2\sqrt{7})yz}{t}\partial_x + t\partial_t\right] \\
X_{11} &= t^{2\sqrt{7}}\left[(1 + \sqrt{7})u\partial_u - (2 + \sqrt{7})z\partial_z - \sqrt{7}y\partial_y + \frac{-tx+2(7+2\sqrt{7})yz}{t}\partial_x - t\partial_t\right] \\
X_{12} &= t^{-2-\sqrt{7}}\left[-(-1 + \sqrt{7})uz\partial_u + (-2 + \sqrt{7})z^2\partial_z - tz\partial_t + (tx + 2yz)\partial_y + \frac{z((-3+\sqrt{7})tx+3yz)}{t}\partial_x\right] \\
X_{13} &= t^{-2+\sqrt{7}}\left[-(1 + \sqrt{7})uz\partial_u + (2 + \sqrt{7})z^2\partial_z + tz\partial_t - (tx + 2yz)\partial_y + t^{-1}z((3 + \sqrt{7})tx - 3yz)\partial_x\right] \\
X_{14} &= \frac{t^2x^2-txyx+3y^2z^2}{t^3}\partial_x + \frac{u(tx+yz)}{t^2}\partial_u + \frac{xz}{t}\partial_z + \frac{y(tx+2yz)}{t^2}\partial_y - \frac{yz}{t}\partial_t
\end{aligned} \tag{2.5}$$

**Case (iv)**  $k(u) = u^n$ ,  $n \neq 0, 1, 3$ . Equation (2.2) in this case admits eight Lie point symmetries given by,

$$\begin{aligned}
X_1 &= t\partial_x \\
X_2 &= 3t^{-1-\sqrt{7}}z\partial_x + (2 + \sqrt{7})t^{-\sqrt{7}}\partial_y \\
X_3 &= 3t^{-1+\sqrt{7}}z\partial_x - (-2 + \sqrt{7})t^{\sqrt{7}}\partial_y \\
X_4 &= -\frac{u}{-1+n}\partial_u + t\partial_t + y\partial_y \\
X_5 &= -\frac{2u}{-1+n}\partial_u + 3t\partial_t + 2y\partial_y - x\partial_x \\
X_6 &= -\frac{2u}{-1+n}\partial_u + 2t\partial_t + y\partial_y + z\partial_z \\
X_7 &= (-2 + \sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_8 &= (2 + \sqrt{7})t^{1+\sqrt{7}}y\partial_x - t^{2+\sqrt{7}}\partial_z
\end{aligned} \tag{2.6}$$

### 2.1.1 Symmetry reduction

We demonstrate the reduction of the (1+3) dimensional wave equation (2.2). The equation with four independent variables is reduced to a partial differential equation that has two independent variables. The reduced equation may then be analysed further using another Lie symmetry reduction or another appropriate method.

Since  $[X, Y] = 0$  with  $X = t\partial_t + y\partial_y$   $Y = t\partial_t - x\partial_x$ , where  $X$  and  $Y$  appear as Lie symmetries (also Noether symmetries as will be seen later) in all the above cases (even not explicitly but as linear combinations), we may begin reducing with  $X$ .

The characteristic equations are

$$\frac{dx}{0} = \frac{dy}{y} = \frac{dz}{0} = \frac{dt}{t} = \frac{du}{0}$$

Integrating yields  $\alpha = \frac{y}{t}$  and (2.2) is reduced to

$$(2x - 6\alpha z)u_{xx} - 6\alpha u_{x\alpha} + 2u_{z\alpha} = 0 \quad (2.7)$$

with  $u = u(x, \alpha, z)$ .

If we then reduce (2.7) by  $Y$ , we obtain the transformation  $\bar{Y} = -\alpha\partial_\alpha - x\partial_x$ . We now have the characteristic equations,

$$\frac{dx}{-x} = \frac{d\alpha}{-\alpha} = \frac{dz}{0} = \frac{du}{0}$$

By integrating, we obtain  $\beta = \frac{x}{\alpha}$  and (2.7) reduces to

$$(4\beta - 3z)u_{\beta\beta} + 3u_\beta - \beta u_{\beta z} = 0 \quad (2.8)$$

with  $u = u(\beta, z)$

Equation (2.8) may be further analysed or reduced using the underlying symmetries. The Lie point symmetries are given by

$$(f_1(\beta, z) + uf_2(z))\partial_u + f_4(z)\partial_z + f_3(\beta, z)\partial_\beta$$

where

$$\begin{aligned} 3zf_3 - 3\beta f_4 - 3z\beta f'_4 + 4\beta^2 f'_4 + \beta^2 f_{3,z} + 3z\beta f_{3,\beta} - 4\beta^2 f_{3,\beta} &= 0, \\ -3f_{1,\beta} + \beta f_{1,\beta z} + 3zf_{1,\beta\beta} - 4\beta f_{1,\beta\beta} &= 0, \\ 12f_3 - 9f_4 - 3z\beta f'_2 + 4\beta^2 f'_2 + 3\beta f_{3,z} + 9zf_{3,\beta} - 12\beta f_{3,\beta} + (3z\beta - 4\beta^2)f_{3,\beta z} + \\ (9z^2 - 24z\beta + 16\beta^2)f_{3,\beta\beta} &= 0 \end{aligned}$$

Take  $f_1 = 0$   $f_2 = 1$ ,  $f_3 = \beta$   $f_4 = z$ , we have the characteristic equations

$$\frac{dz}{z} = \frac{d\beta}{\beta} = \frac{du}{u}$$

By integrating we obtain  $\gamma = \frac{\beta}{z}$ ,  $u = \frac{U}{z}$ ,  $U = U(\gamma)$  and (2.8) reduces to the second order ODE

$$(\gamma^2 + 4\gamma - 3)U_{\gamma\gamma} + (2\gamma + 3)U_\gamma = 0.$$

The solution is

$$U(\gamma) = C_1 + \exp \left[ -\frac{1}{7} \operatorname{arctanh} \left( \frac{1}{7}(x+2)\sqrt{7} \right) \right] C_2$$

where  $C_1, C_2$  are constant

## 2.2 Noether symmetries

Consider the wave equation (2.2). Since it is variational, the corresponding Lagrangian is given by

$$L = \frac{tx - 3yz}{t^2} u_x^2 - \frac{2y}{t} u_y u_x + u_t u_x + u_z u_y - 2h(u) \quad (2.9)$$

where  $h(u) = \int k(u) du$

**Case (i)**  $h(u) = 0$

Let

$$X = \xi(x, y, z, t, u) \partial_x + \eta(x, y, z, t, u) \partial_y + \gamma(x, y, z, t, u) \partial_z + \tau(x, y, z, t, u) \partial_t + \phi(x, y, z, t, u) \partial_u$$

be a Noether point operator with gauge vector  $f_i (i = 1, 2, 3, 4)$  dependent on  $(t, x, y, z, u)$ . This becomes, for Lagrangian given by (2.9),

$$XL + L[D_x \xi + D_y \eta + D_z \gamma + D_t \tau] = D_x f_1 + D_y f_2 + D_z f_3 + D_t f_4$$

Separation by derivatives of  $u$  yields the following overdetermined system



$$\begin{aligned}
u_x^3 &: \xi_u \\
u_x^2 u_y &: \eta_u \\
u_x^2 u_t &: \tau_u \\
u_x u_t u_z &: \gamma_u \\
u_x u_y &: -2yt^2 \tau_t - 4t^2 y \phi_u - t(2xt - 6yz) \eta_x - \eta_t t^3 - 2t^2 y \gamma_z - t^3 \xi_z + 2yt \tau - 2t^2 \eta \\
u_x u_z &: -t(2xt - 6yz) \gamma_x - \gamma_t t^3 + 2t^2 y \gamma_y - t^3 \xi_y \\
u_x u_t &: 2t^3 \phi_u - t(2xt - 6yz) \tau_x + 2t^2 y \tau_y + t^3 \gamma_z + t^3 \eta_y \\
u_y u_z &: 2\gamma_x t^2 y + 2t^3 \phi_u + \tau_t t^3 + \xi_x t^3 \\
u_y u_t &: -\eta_x t^3 + 2\tau_x t^2 y - \tau_z t^3 \\
u_z u_t &: -\gamma_x t^3 - \tau_y t^3 \\
u_x^2 &: (xt - 3yz)(2t\phi_u + t\tau_t + t\gamma_z + t\eta_y - t\xi_x) + 2t^2 y \xi_y - t^3 \xi_t - (xt - 6yz)\tau + (-3z\eta + t\xi - 3y\gamma)t \\
u_y^2 &: 2y\eta_x - \eta_z t \\
u_z^2 &: -\gamma_y \\
u_t^2 &: \tau_x \\
u_z &: \phi_y - f_{3,u} \\
u_x &: t(2xt - 6yz)\phi_x + \phi_t t^3 - f_{1,u} t^3 - 2t^2 y \phi_y \\
u_t &: \phi_x - f_{4,u} \\
u_y &: -2y\phi_x + \phi_z t - f_{2,u} t \\
1 &: f_{1,x} + f_{2,y} + f_{3,z} + f_{4,z}
\end{aligned}
\tag{2.10}$$

The detailed calculations lead to

$$\begin{aligned}
X_1 &= x\partial_x + y\partial_y - \frac{u}{2}\partial_u \\
X_2 &= -x\partial_x + t\partial_t \\
X_3 &= x\partial_x + z\partial_z - \frac{u}{2}\partial_u \\
X_4 &= t\sqrt{7} \left[ -\frac{z(2+\sqrt{7})}{t}\partial_x + \partial_y \right] \\
X_5 &= \frac{1}{t\sqrt{7}} \left[ \frac{z(-2+\sqrt{7})}{t}\partial_x + \partial_y \right] \\
X_6 &= -t^{1+\sqrt{7}}y(2+\sqrt{7})\partial_x + t^{2+\sqrt{7}}\partial_z \\
X_7 &= yt^{1-\sqrt{7}}(-2+\sqrt{7})\partial_x + t^{2-\sqrt{7}}\partial_z \\
X_8 &= t\partial_x \\
X_9 &= \frac{t^2x^2-txyz+3y^2z^2}{2t^3}\partial_x + \frac{y(tx+2yz)}{2t^2}\partial_y + \frac{xz}{2t}\partial_z - \frac{yz}{2t}\partial_t - \frac{u(tx+yz)}{2t^2}\partial_u \\
X_{10} &= t\sqrt{7} \left[ -\frac{z(\sqrt{7}tx+3xt-3yz)}{2t^3}\partial_x + \frac{tx+2yz}{2t^2}\partial_y - \frac{z^2(2+\sqrt{7})}{2t^2}\partial_z - \frac{z}{2t}\partial_t + \frac{zu(1+\sqrt{7})}{2t^2}\partial_u \right] \\
X_{11} &= \frac{1}{t\sqrt{7}} \left[ \frac{z(\sqrt{7}tx-3xt+3yz)}{2t^3}\partial_x + \frac{tx+2yz}{2t^2}\partial_y + \frac{z^2(-2+\sqrt{7})}{2t^2}\partial_z - \frac{z}{2t}\partial_t - \frac{zu(-1+\sqrt{7})}{2t^2}\partial_u \right] \\
X_{12} &= t\sqrt{7} \left[ \frac{y(3\sqrt{7}tx-3\sqrt{7}yz+7xt)}{14t}\partial_x + \frac{y^2}{2}\partial_y - \frac{\sqrt{7}tx}{14}\partial_z + \frac{\sqrt{7}ty}{14}\partial_t - \frac{yu(\sqrt{7}+7)}{14}\partial_u \right] \\
X_{13} &= \frac{1}{t\sqrt{7}} \left[ \frac{-y(3\sqrt{7}tx-3\sqrt{7}yz-7xt)}{14t}\partial_x + \frac{y^2}{2}\partial_y + \frac{\sqrt{7}tx}{14}\partial_z - \frac{\sqrt{7}ty}{14}\partial_t + \frac{yu(\sqrt{7}-7)}{14}\partial_u \right] \\
X_{14} &= t^{2\sqrt{7}} \left[ \frac{\sqrt{7}tx-14\sqrt{7}yz-28yz}{7t}\partial_x + y\partial_y + \frac{z(7+2\sqrt{7})}{7}\partial_z + \frac{\sqrt{7}t}{7}\partial_t - \frac{7+\sqrt{7}}{7}u\partial_u \right] \\
X_{15} &= \frac{1}{t^{2\sqrt{7}}} \left[ -\frac{\sqrt{7}tx-14\sqrt{7}yz+28yz}{7t}\partial_x + y\partial_y - \frac{z(-7+2\sqrt{7})}{7}\partial_z - \frac{\sqrt{7}t}{7}\partial_t + \frac{-7+\sqrt{7}}{7}u\partial_u \right]
\end{aligned} \tag{2.11}$$

The conserved forms corresponding to each Noether symmetry ([13]) is a three form  $\omega$  such that the four form  $D\omega$  vanishes. Thus

$$\omega = \Phi^x dy \wedge dz \wedge dt - \Phi^y dx \wedge dz \wedge dt + \Phi^z dx \wedge dy \wedge dt - \Phi^t dx \wedge dy \wedge dz$$

so that

$$D_x\Phi^x + D_y\Phi^y + D_z\Phi^z + D_t\Phi^t = 0$$

We list all conserved flux for one case, and only the conserved densities for the

remaining symmetries.

$$\begin{aligned}
\Phi_1^x &= \frac{1}{t^2}(2t^2xu_yu_z - 2t^2yu_tu_y - 2tx^2u_x^2 - 4txyu_xu_y + 4ty^2u_y^2 + 6xyzu_x^2 + 12y^2zu_xu_y - t^2uu_t - \\
&\quad 2tuxu_x + 2tyuu_y + 6wyzu_x) \\
\Phi_1^y &= \frac{1}{t^2}(2t^2xu_xu_z - 2t^2yu_tu_x - 6txyu_x^2 + 6y^2zu_x^2 + t^2uu_z - 2tuyu_x) \\
\Phi_1^z &= u_y(2xu_x + 2yu_y + u) \\
\Phi_1^t &= u_x(2xu_x + 2yu_y + u) \\
\Phi_2^t &= \frac{1}{t}(t^2u_yu_z + 2txu_x^2 - 2tyu_xu_y - 3yzu_x^2) \\
\Phi_3^t &= -\frac{u_x}{2}(2xu_x + 2zu_z + u) \\
\Phi_4^t &= u_x(u_xt^{-1+\sqrt{7}}z\sqrt{7} + 2u_xt^{-1+\sqrt{7}} - u_yt^{\sqrt{7}}) \\
\Phi_5^t &= -u_x(u_xt^{-1-\sqrt{7}}z\sqrt{7} - 2u_xt^{-1-\sqrt{7}} + u_yt^{-\sqrt{7}}) \\
\Phi_6^t &= u_x(u_xt^{1+\sqrt{7}}y\sqrt{7} + 2u_xt^{1+\sqrt{7}}y - u_zt^{2+\sqrt{7}}) \\
\Phi_7^t &= -u_x(u_xt^{1-\sqrt{7}}y\sqrt{7} - 2u_xt^{1-\sqrt{7}}y + u_zt^{2-\sqrt{7}}) \\
\Phi_8^t &= -u_x^2t \\
\Phi_9^t &= -\frac{1}{2t^2}(tx^2u_x^2 + txyu_xu_y + txzu_xu_z + tyzu_yu_z + tuxu_x + yzu_x) + \frac{u}{2t} \\
\Phi_{10}^t &= \frac{1}{-2}t^{-1+\sqrt{7}}zu_yu_z + t^{-2+\sqrt{7}}xzu_x^2 + \frac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}xzu_x^2 + \frac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}z^2u_xu_z + \frac{1}{2}\sqrt{7}t^{-2+\sqrt{7}}zuu_x + \\
&\quad t^{-2+\sqrt{7}}z^2u_xu_z - \frac{1}{2}t^{-1+\sqrt{7}}xu_xu_y + \frac{1}{2}t^{-2+\sqrt{7}}zuu_x \\
\Phi_{11}^t &= \frac{1}{2}\left(-t^{-1-\sqrt{7}}zu_yu_z + 2t^{-2-\sqrt{7}}xzu_x^2 - t^{-2-\sqrt{7}}\sqrt{7}xzu_x^2 - t^{-2-\sqrt{7}}\sqrt{7}z^2u_xu_z - \right. \\
&\quad \left. t^{-2-\sqrt{7}}\sqrt{7}zuu_x + 2t^{-2-\sqrt{7}}z^2u_xu_z + t^{-2-\sqrt{7}}zuu_x - t^{-1-\sqrt{7}}xu_xu_y\right) \\
\Phi_{12}^t &= \frac{2}{7}\left(\sqrt{7}t^{1+\sqrt{7}}yu_yu_z - 2t^{\sqrt{7}}\sqrt{7}xyu_x^2 - 2\sqrt{7}t^{\sqrt{7}}y^2u_xu_y + t^{1+\sqrt{7}}\sqrt{7}xu_xu_z - t^{\sqrt{7}}\sqrt{7}yu_xu_x \right. \\
&\quad \left. - 14t^{\sqrt{7}}xyu_x^2 - 14t^{\sqrt{7}}y^2u_xu_y - 14t^{\sqrt{7}}yu_xu_x\right) \\
\Phi_{13}^t &= \frac{1}{14}\left(-t^{1-\sqrt{7}}\sqrt{7}yu_yu_z + 2t^{-\sqrt{7}}\sqrt{7}xyu_x^2 + 2\sqrt{7}t^{-\sqrt{7}}y^2u_xu_y + t^{-\sqrt{7}}\sqrt{7}yu_xu_x - 7t^{-\sqrt{7}}xyu_x^2 - \right. \\
&\quad \left. 7t^{-\sqrt{7}}y^2u_xu_y - t^{1-\sqrt{7}}\sqrt{7}xu_xu_z - 7t^{-\sqrt{7}}yu_xu_x\right) \\
\Phi_{14}^t &= \frac{1}{7}\left(\sqrt{7}t^{2\sqrt{7}+1}u_yu_z - 2\sqrt{7}t^{2\sqrt{7}}yu_xu_y + 11t^{2\sqrt{7}-1}\sqrt{7}yzu_x^2 - 2t^{2\sqrt{7}}\sqrt{7}zu_xu_z + 28t^{2\sqrt{7}-1}yzu_x^2 - \right. \\
&\quad \left. 7t^{2\sqrt{7}}\sqrt{7}uu_x - 7t^{2\sqrt{7}}yu_xu_y - 7t^{2\sqrt{7}}zu_xu_z - 7t^{2\sqrt{7}}uu_x\right) \\
\Phi_{15}^t &= -\frac{1}{7}\left(\sqrt{7}t^{1-2\sqrt{7}}u_yu_z - 2\sqrt{7}t^{-2\sqrt{7}}yu_xu_y + 11t^{-2\sqrt{7}-1}\sqrt{7}yzu_x^2 - 2t^{-2\sqrt{7}}\sqrt{7}zu_xu_z - 28t^{-2\sqrt{7}-1}yzu_x^2 \right. \\
&\quad \left. - t^{-2\sqrt{7}}\sqrt{7}u_x + 7t^{-2\sqrt{7}}yu_xu_y + 7t^{2\sqrt{7}}zu_xu_z - 7t^{-2\sqrt{7}}u_x\right)
\end{aligned} \tag{2.12}$$

**Case (ii)**  $h(u) = u^n$ ,  $n \neq 0, 1$

It can be shown that a 7-dimensional algebra of point symmetry generators with basis (Noether symmetries) is given by

$$\begin{aligned}
X_1 &= -x\partial_x + t\partial_t, & X_2 &= -y\partial_y + z\partial_z & X_3 &= -(2 + \sqrt{7})t^{1+\sqrt{7}}y\partial_x + t^{2+\sqrt{7}}\partial_z \\
X_4 &= (-2 + \sqrt{7})t^{1-\sqrt{7}}y\partial_x + t^{2-\sqrt{7}}\partial_z & X_5 &= -(2 + \sqrt{7})t^{-1+\sqrt{7}}z\partial_x + t^{\sqrt{7}}\partial_y \\
X_6 &= (-2 + \sqrt{7})t^{-1-\sqrt{7}}z\partial_x + t^{-\sqrt{7}}\partial_y & X_7 &= t\partial_x
\end{aligned} \tag{2.13}$$

The corresponding conserved quantities are

$$\begin{aligned}
\Phi_1^t &= \frac{-u_y u_z t^2 - 2txu_x^2 + 2yu_x u_y t + 3yzu_x^2 + u^n t^2}{t} \\
\Phi_1^z &= u_y (tu_t - xu_x) \\
\Phi_1^y &= -\frac{(tu_z - 2yu_x)(tu_t - xu_x)}{t} \\
\Phi_1^x &= -\frac{-t^3 u_t^2 - 2t^2 xu_x u_t - t^2 xu_y u_z + 2t^2 yu_y u_t + tx^2 u_x^2 + 6tyzu_x u_t - 3xyz u_x^2 + u^n t^2 x}{t^2} \\
\Phi_2^t &= u_x (yu_y - zu_z) \\
\Phi_2^z &= \frac{-t^2 yu_y^2 - t^2 zu_x u_t - txzu_x^2 + 2tyzu_x u_y + 3yz^2 u_x^2 + u^n t^2 z}{t^2} \\
\Phi_2^y &= \frac{-t^2 yu_x u_t - t^2 zu_z^2 - txyu_x^2 + 2tyzu_x u_z + 3y^2 zu_x^2 + u^n t^2 y}{t^2} \\
\Phi_2^x &= -\frac{(t^2 u_t + 2txu_x - 2tyu_y - 6yzu_x)(yu_y - zu_z)}{t^2} \\
\Phi_3^t &= -u_x \left( -u_x t^{1+\sqrt{7}} y \sqrt{7} - 2u_x t^{1+\sqrt{7}} y + u_z t^{2+\sqrt{7}} \right) \\
\Phi_3^z &= (3t^{\sqrt{7}} y z u_x^2 - \sqrt{7} t^{1+\sqrt{7}} y u_x u_y - t^{1+\sqrt{7}} x u_x^2 - t^{2+\sqrt{7}} u_x u_t + t^{2+\sqrt{7}} u^n) \\
\Phi_3^y &= -\frac{(tu_z - 2yu_x)(-u_x t^{1+\sqrt{7}} y \sqrt{7} - 2u_x t^{1+\sqrt{7}} y + u_z t^{2+\sqrt{7}})}{t} \\
\Phi_3^x &= \frac{1}{t^2} \left( 3\sqrt{7} t^{1+\sqrt{7}} y^2 z u_x^2 + 6t^{1+\sqrt{7}} y^2 z u_x^2 - t^{2+\sqrt{7}} \sqrt{7} x y u_x^2 - 2t^{2+\sqrt{7}} x y u_x^2 + \sqrt{7} t^{3+\sqrt{7}} y u_y u_z - \right. \\
&\quad \left. \sqrt{7} u^n t^{3+\sqrt{7}} y + t^{4+\sqrt{7}} u_z u_t + 2t^{3+\sqrt{7}} x u_x u_z - 6t^{2+\sqrt{7}} y z u_x u_z - 2u^n t^{3+\sqrt{7}} y \right)
\end{aligned}$$

$$\begin{aligned}
\Phi_4^t &= -u_x \left( u_x t^{1-\sqrt{7}} y \sqrt{7} - 2u_x t^{1-\sqrt{7}} y + u_z t^{2-\sqrt{7}} \right) \\
\Phi_4^z &= (t^{1-\sqrt{7}} \sqrt{7} y u_x u_y + 3t^{1-\sqrt{7}} y z u_x^2 - t^{1-\sqrt{7}} x u_x^2 - t^{2-\sqrt{7}} u_x u_t + t^{2-\sqrt{7}} u^n) \\
\Phi_4^y &= -\frac{(tu_z - 2yu_x)(u_x t^{1-\sqrt{7}} y \sqrt{7} - 2u_x t^{1-\sqrt{7}} y + u_z t^{2-\sqrt{7}})}{t} \\
\Phi_4^x &= -\frac{1}{t^2} \left( 3t^{1-\sqrt{7}} \sqrt{7} y^2 z u_x^2 - 6t^{1-\sqrt{7}} y^2 z u_x^2 - t^{2-\sqrt{7}} \sqrt{7} x y u_x^2 + t^{3-\sqrt{7}} \sqrt{7} y u_y u_z + 2t^{2-\sqrt{7}} x y u_x^2 - \right. \\
&\quad \left. t^{3-\sqrt{7}} \sqrt{7} u^n y + 2t^{3-\sqrt{7}} u^n y - t^{4-\sqrt{7}} u_z u_t - 2t^{3-\sqrt{7}} x u_x u_z + 6t^{2-\sqrt{7}} y z u_x u_z \right) \\
\Phi_5^t &= u_x \left( u_x t^{-1+\sqrt{7}} z \sqrt{7} + 2u_x t^{-1+\sqrt{7}} z - u_y t^{\sqrt{7}} \right) \\
\Phi_5^z &= -u_y \left( u_x t^{-1+\sqrt{7}} z \sqrt{7} + 2u_x t^{-1+\sqrt{7}} z - u_y t^{\sqrt{7}} \right) \\
\Phi_5^y &= -\frac{-t^{\sqrt{7}} \sqrt{7} z u_x u_z + 2t^{-1+\sqrt{7}} \sqrt{7} y z u_x^2 + 7t^{-1+\sqrt{7}} y z u_x^2 - t^{\sqrt{7}} x u_x^2 - 2t^{\sqrt{7}} z u_x u_z - t^{1+\sqrt{7}} u_x u_t + t^{1+\sqrt{7}} u^n}{t} \\
\Phi_5^x &= \frac{1}{t^2} \left[ 3t^{-1+\sqrt{7}} \sqrt{7} y z^2 u_x^2 - \sqrt{7} t^{\sqrt{7}} x z u_x^2 + 6t^{-1+\sqrt{7}} y z^2 u_x^2 + t^{1+\sqrt{7}} \sqrt{7} z u_y u_z - 2t^{\sqrt{7}} x z u_x^2 - \right. \\
&\quad \left. 6t^{\sqrt{7}} y z u_x u_y - t^{1+\sqrt{7}} \sqrt{7} u^n z + 2t^{1+\sqrt{7}} z u_y u_z - 2t^{1+\sqrt{7}} u^n z + t^{2+\sqrt{7}} u_y u_t + 2t^{1+\sqrt{7}} x u_x u_y \right. \\
&\quad \left. - 2t^{1+\sqrt{7}} y u_y^2 \right] \\
\Phi_6^t &= -u_x \left( u_x t^{-1-\sqrt{7}} z \sqrt{7} - 2u_x t^{-1-\sqrt{7}} z + u_y t^{-\sqrt{7}} \right) \\
\Phi_6^z &= u_y \left( u_x t^{-1-\sqrt{7}} z \sqrt{7} - 2u_x t^{-1-\sqrt{7}} z + u_y t^{-\sqrt{7}} \right) \\
\Phi_6^y &= \frac{-7t^{-1-\sqrt{7}} y z u_x^2 - t^{-\sqrt{7}} \sqrt{7} z u_x u_z + 2t^{-1-\sqrt{7}} \sqrt{7} y z u_x^2 + t^{-\sqrt{7}} x u_x^2 + 2t^{-\sqrt{7}} z u_x u_z + t^{1-\sqrt{7}} u_x u_t - t^{1-\sqrt{7}} u^n}{t}
\end{aligned} \tag{2.14}$$

$$\begin{aligned}
\Phi_6^x &= -\frac{1}{t^2} \left( 3t^{-1-\sqrt{7}} \sqrt{7} y z^2 u_x^2 - t^{-\sqrt{7}} \sqrt{7} x z u_x^2 - 6t^{-1-\sqrt{7}} y z^2 u_x^2 + 2t^{-\sqrt{7}} x z u_x^2 + 6t^{-\sqrt{7}} y z u_x u_y + \right. \\
&\quad \left. t^{1-\sqrt{7}} \sqrt{7} z u_y u_z - t^{1-\sqrt{7}} \sqrt{7} u^n z - 2t^{1-\sqrt{7}} z u_y u_z + 2t^{1-\sqrt{7}} u^n z - t^{2-\sqrt{7}} u_y u_t - 2t^{1-\sqrt{7}} x u_x u_y + \right. \\
&\quad \left. 2t^{1-\sqrt{7}} y u_y^2 \right) \\
\Phi_7^t &= -u_x^2 t \\
\Phi_7^z &= +u_y t u_x \\
\Phi_7^y &= -(tu_z - 2yu_x) u_x \\
\Phi_7^x &= \frac{-u_y u_z t^2 + t x u_x^2 - 3y z u_x^2 + u^n t^2}{t}
\end{aligned} \tag{2.15}$$

### 2.3 Variational symmetries, multipliers approach

Consider the wave equation (2.2) in ASD-Einstein spacetime with dependent variable  $u = u(x, y, t)$ , i.e., we have taken out the spatial variable  $z$  from the original equation because the calculations with  $z$  are extremely cumbersome producing no final outcomes. We consider the multiplier method for (2.2). Let us choose  $k(u) = -u$ , we have

$$\frac{\delta}{\delta u} \left[ \mathcal{Q} \left( \frac{x}{t} u_{xx} - \frac{2y}{t} u_{xy} + u_{x,t} + u \right) \right] = 0$$

where  $\mathcal{Q} = \mathcal{Q}(x, y, u, u_x, u_y, u_{x,x}, u_{y,y}, u_{x,y}, u_{x,x,x}, u_{x,x,y})$ . Even though not pursued here, the calculation may include  $t$  and derivatives of  $u$  with respect to  $t$ . Then

$$\frac{\delta}{\delta u} \left[ \mathcal{Q} \left( \frac{x}{t} u_{xx} - \frac{2y}{t} u_{xy} + u_{x,t} + u \right) \right] = D_t \Phi^t + D_y \Phi^y + D_x \Phi^x$$

where  $(\Phi^x, \Phi^y, \Phi^t)$  is the conserved flow. We obtain the set of multiplier  $\mathcal{Q}_i$  together with the conserved densities  $\Phi_i^t$ , namely,

$$\begin{aligned} \mathcal{Q}_1 &= \frac{u_x}{\sqrt{y}} \\ \Phi_1^t &= \frac{u_x^2 - u u_{xx}}{4\sqrt{y}} \\ \mathcal{Q}_2 &= \frac{1}{ty} (t u_x + x u_{xxx} + u_{xx}) \\ \Phi_2^t &= \frac{1}{4ty} [t u_x^2 + (u_{xx} + x u_{xxx}) u_x - u (t u_{xx} + 2 u_{xxx} + x u_{xxxx})] \\ \mathcal{Q}_3 &= y u_y + \frac{1}{2} u \\ \Phi_3^t &= \frac{1}{4} y (u_y u_x - u u_{xy}) \end{aligned} \tag{2.16}$$

$$\begin{aligned}
\mathcal{Q}_4 &= \frac{1}{4t\sqrt{y}}(4tyu_y + 4xyu_{xy} + tu + xu_{xx} + 2yu_{xy}) \\
\Phi_t^4 &= \frac{1}{16t\sqrt{y}}[4tyu_x + (2yu_{xy} + xu_{xx} + 4xyu_{xy})u_x - u(4tyu_y + u_{xx} + 6yu_{xty} + xu_{xxx} + 4xyu_{xxy})] \\
\mathcal{Q}_5 &= \frac{u_{xxx}}{y^{2/3}} \\
\Phi_t^6 &= \frac{1}{4}(u_x u_{xy} - u u_{xxy})
\end{aligned}
\tag{2.17}$$

$$\tag{2.18}$$

### 3 Wave equations on Kerr spacetime

#### 3.1 Introduction

In 1963, R. P. Kerr proposed a metric that describes a massive rotating object. Since then, a huge number of papers about the structure and astrophysical applications of this spacetime appeared. Several investigations in the literature have been aimed at that spacetime. Exact symmetries of the Kerr spacetime are given in [11]. In [6], the authors investigated on the approximate symmetries on Kerr spacetime and find the rescaling factor for the energy. In this section, we analyse the symmetry structure of wave equation on Kerr spacetime. Noether approach and direct construction are used to find conserved densities on this spacetime.

The line element in this spacetime is given by [10]

$$ds^2 = \frac{\Delta}{\rho^2} [dt - k \sin^2 \theta d\phi]^2 - \frac{\sin^2 \theta}{\rho^2} [(r^2 + k^2)d\phi - kdt]^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 \tag{3.19}$$

where  $\Delta = r^2 - 2Mr + k^2$  and  $\rho^2 = r^2 + a^2 \cos^2 \theta$ .  $M$  and  $k$  represent the mass and the rotation parameter, respectively. The angular momentum of the object is  $J = Mk$ . The Gordon type equation is given by (2.1)

$$\begin{aligned}
0 = & \frac{1}{(2Mr-k^2-r^2)} \left[ 4 \left( Mr - \frac{k^2}{2} - \frac{r^2}{2} \right)^2 \sin^2 \theta u_{r,r} - 2 \left( Mr - \frac{k^2}{2} - \frac{r^2}{2} \right) \sin^2 \theta u_{\theta,\theta} \right. \\
& + 2 \sin^2 \theta \left( k^2 \left( Mr - \frac{k^2}{2} - \frac{r^2}{2} \right) \cos^2 \theta - Mk^2 r - \frac{k^2 r^2}{2} - \frac{r^4}{2} \right) u_{t,t} + (k^2 \cos^2 \theta - 2Mr + r^2) u_{\phi,\phi} - \\
& \left. 4 \sin \theta (Mu_{t\phi} k r \sin \theta - (\sin \theta (M-r) u_r - \frac{1}{2} u_\theta \cos \theta) \left( Mr - \frac{k^2}{2} + \frac{r^2}{2} \right)) \right] - k(u)(r^2 + k^2 \cos^2 \theta) \sin^2 \theta
\end{aligned} \tag{3.20}$$

The corresponding Lagrangian is

$$\begin{aligned}
L = & \frac{1}{2Mr-k^2-r^2} \left[ \left( k^2 \sin \theta (\cos \theta)^2 \left( Mr - \frac{1}{2} k^2 - \frac{1}{2} r^2 \right) - Mk^2 r \sin \theta - \frac{1}{2} k^2 r^2 \sin \theta - \frac{1}{2} \sin \theta r^4 \right) u_t^2 \right] \\
& - 2 \frac{kMr \sin \theta u_t u_\phi}{2Mr-k^2-r^2} + \frac{1}{2} \sin \theta (2Mr - k^2 - r^2) u_r^2 - \frac{1}{2} \sin \theta u_\theta^2 + \\
& \frac{1}{2} \frac{(k^2 (\cos \theta)^2 - 2Mr + r^2) u_\phi^2}{\sin \theta (2Mr - k^2 - r^2)} - h(u) \sin \theta (r^2 + k^2 \cos \theta)
\end{aligned} \tag{3.21}$$

where  $h(u) = \int k(u) du$

### 3.2 Symmetries of the waves equation-the Noether approach

Many of the calculation have been left out as they are tedious. We classify the cases that yield strict Noether symmetries (zero gauge) of (3.20)

The principal Noether algebra is for the case  $h(u) = 0$  in (3.21)

$$\begin{aligned}
X_1 &= \partial_u \\
X_3 &= \partial_t \\
X_2 &= \partial_\phi
\end{aligned} \tag{3.22}$$

The associated conserved vectors are



$$\begin{aligned}
\Phi_1^\phi &= \frac{-2kMr u_t + 2M(\cos(\theta))^2 k r u_t - 2Mr u_\phi + (\cos(\theta))^2 k^2 u_\phi + r^2 u_\phi}{\sin(\theta)(2Mr - k^2 - r^2)} \\
\Phi_1^\theta &= \sin \theta u_\theta \\
\Phi_1^r &= (2Mr - k^2 - r^2) u_r \sin(\theta) \\
\Phi_1^t &= -\frac{\sin \theta (2M(\cos \theta)^2 k^2 r u_t - (\cos \theta)^2 k^4 u_t - (\cos \theta)^2 k^2 r^2 u_t - 2Mk^2 r u_t - k^2 r^2 u_t - u_t r^4 - 2kMr u_\phi)}{2Mr - k^2 - r^2} \\
\Phi_2^t &= \frac{\sin \theta (2M(\cos \theta)^2 k^2 r u_t - (\cos \theta)^2 k^4 u_t - (\cos \theta)^2 k^2 r^2 u_t - 2Mk^2 r u_t - k^2 r^2 u_t - u_t r^4 - 2kMr u_\phi) u_\phi}{2Mr - k^2 - r^2} \\
\Phi_3^t &= \frac{1}{2\sin \theta (2Mr - k^2 - r^2)} \left[ -4M^2 r^2 u_r^2 - 4M(\cos \theta)^2 k^2 r u_r^2 - 2M(\cos \theta)^4 k^2 r u_t^2 + \right. \\
&\quad 4M(\cos \theta)^2 k^2 r u_t^2 - k^2 u_\theta^2 - r^2 u_\theta^2 - r^4 u_t^2 - r^4 u_r^2 - k^4 u_r^2 - r^2 u_\phi^2 - 2Mk^2 r u_t^2 + \\
&\quad (\cos \theta)^4 k^2 r^2 u_t^2 + 4M^2(\cos \theta)^2 r^2 u_r^2 - 4M(\cos \theta)^2 r^3 u_r^2 + 2(\cos \theta)^2 k^2 r^2 u_r^2 - \\
&\quad 2M(\cos \theta)^2 r u_\theta^2 + 4Mk^2 r u_r^2 + 2Mr u_\theta^2 - 2k^2 r^2 u_r^2 + 4Mr^3 u_r^2 + (\cos \theta)^4 k^4 u_t^2 + \\
&\quad (\cos \theta)^2 k^4 u_r^2 + (\cos \theta)^2 r^4 u_t^2 + (\cos \theta)^2 r^4 u_r^2 + (\cos \theta)^2 k^2 u_\theta^2 + (\cos \theta)^2 r^2 u_\theta^2 - \\
&\quad \left. (\cos \theta)^2 k^4 u_t^2 + 2Mr u_\phi^2 - (\cos \theta)^2 k^2 u_\phi^2 - k^2 r^2 u_t^2 \right]
\end{aligned} \tag{3.23}$$

### 3.3 Symmetries of the waves equation-the multipliers approach

Consider the wave equation (3.20) with  $k(u) = 0$ . We have

$$\begin{aligned}
\frac{\delta}{\delta u}[\mathcal{Q}] &= \frac{1}{(2Mr - k^2 - r^2)} \left[ 4 \left( Mr - \frac{k^2}{2} - \frac{r^2}{2} \right)^2 \sin^2 \theta u_{rr} - 2 \left( Mr - \frac{k^2}{2} - \frac{r^2}{2} \right) \sin^2 \theta u_{\theta\theta} \right. \\
&\quad + 2 \sin^2 \theta \left( k^2 \left( Mr - \frac{k^2}{2} - \frac{r^2}{2} \right) \cos^2 \theta - Mk^2 r - \frac{k^2 r^2}{2} - \frac{r^4}{2} \right) u_{t,t} + (k^2 \cos^2 \theta - 2Mr + r^2) u_{\phi\phi} - \\
&\quad \left. 4 \sin \theta (Mu_{t\phi} k r \sin \theta - (\sin \theta (M - r) u_r - \frac{1}{2} u_\theta \cos \theta) \left( Mr - \frac{k^2}{2} + \frac{r^2}{2} \right)) \right] \\
&= D_t \Phi^t + D_r \Phi^r + D_\theta \Phi^\theta + D_\phi \Phi^\phi
\end{aligned} \tag{3.24}$$

where  $\mathcal{Q} = \mathcal{Q}(u_\phi, u_t, u_{\phi,\phi}, u_{t,t}, u_{t,\phi}, u_{\phi,\phi,t}, u_{t,\phi,t}, u_{\phi,\phi,\phi})$

After tedious calculations, we obtain a set of multipliers  $\mathcal{Q}_i$  together with the conserved densities.

$$\begin{aligned}
\mathcal{Q}_1 &= u_t \\
\Phi_1^t &= \frac{1}{2(k^2+r(r-2M))\sin(\theta)} [\sin(\theta)^2 u_t (2kMr u_\phi + ((k^2 + r(r-2M)) \cos(\theta) k^2 + r(r^3 + k^2(2M+r))) u_t) - \\
&\quad u(2k^2 \cos(\theta)^2 + (k^2 + r(r-2M)) \sin(\theta) u_\theta \cos(\theta) + 2r(r-2M) + \sin(\theta)^2 (u_{rr} k^4 + 2r^2 u_{rr} k^2 - \\
&\quad 4Mr u_{rr} k^2 - 2Mr u_{t\phi} k + (k^2 + r(r-2M)) u_{\theta\theta} - 2(M-r)(k^2 + r^2 - 2Mr) u_r + r^4 u_{rr} - \\
&\quad 4Mr^3 u_{rr} + 4M^2 r^2 u_{rr}))] \\
\mathcal{Q}_2 &= 1 \\
\Phi_2^t &= \frac{1}{k^2+r(r-2M)} [\sin(\theta) (2kMr u_\phi + ((k^2 + r(r-2M)) \cos(\theta) k^2 + r(r^3 + k^2(2M+r))) u_t)] \\
\mathcal{Q}_3 &= u_\phi \\
\Phi_3^t &= \frac{1}{2(k^2+r(r-2M))} [\sin(\theta) (2kMr u_\phi^2 + ((k^2 + r(r-2M)) \cos(\theta) k^2 + r(r^3 + k^2(2M+r))) u_t u_\phi - \\
&\quad u(2kMr u_{\phi\phi} + ((k^2 + r(r-2M)) \cos(\theta) k^2 + r(r^3 + k^2(2M+r))) u_{t\phi}))] \\
\mathcal{Q}_4 &= u_{\phi\phi\phi} \\
\Phi_4^t &= \frac{1}{2(k^2+r(r-2M))} [\sin(\theta) (2kMr u_\phi u_{\phi\phi\phi} + ((k^2 + r(r-2M)) \cos(\theta) k^2 + r(r^3 + k^2(2M+r))) u_t u_{\phi\phi\phi} - \\
&\quad u(2kMr u_{\phi\phi\phi\phi} + ((k^2 + r(r-2M)) \cos(\theta) k^2 + r(r^3 + k^2(2M+r))) u_{t\phi\phi\phi}))] \\
&\hspace{15em} (3.25)
\end{aligned}$$

$$\begin{aligned}
\mathcal{Q}_5 &= u_{tt\phi} \\
\Phi_5^t &= \frac{1}{6(k^2+r(r-2M))\sin(\theta)} [-4k^2u_{t\phi}\cos(\theta)^2 + (k^2+r(r-2M))\sin(\theta)(2\sin(\theta)u_{t\phi}u_{tt}k^2 + \\
& 2\sin(\theta)u_{tu}u_{tt\phi}k^2 - \sin(\theta)u_{\phi u}u_{ttt}k^2 - \sin(\theta)uu_{ttt\phi}k^2 + u_{\theta\phi}u_t - 2u_{\theta u}u_{t\phi} + u_{\phi u}u_{t\theta} - \\
& 2uu_{t\theta\phi})\cos(\theta) + 4(2M-r)ru_{t\phi} + \sin(\theta)^2(u_{rr\phi}u_tk^4 - 2u_{rr}u_{t\phi}k^4 + u_{\phi u}u_{trr}k^4 - 2uu_{trr\phi}k^4 + \\
& 2r^2u_{rr\phi}u_tk^2 - 4Mr u_{rr\phi}u_tk^2 - 2u_{\theta\theta}u_{t\phi}k^2 + 4Mu_{ru}u_{t\phi}k^2 - 4ru_{ru}u_{t\phi}k^2 - 4r^2u_{rr}u_{t\phi}k^2 + \\
& 8Mr u_{rr}u_{t\phi}k^2 + u_{\phi u}u_{t\theta\theta}k^2 - 2uu_{t\theta\theta\phi}k^2 - 2Mu_{\phi u}u_{tr}k^2 + 2ru_{\phi u}u_{tr}k^2 + 4Mu_{uu}u_{tr\phi}k^2 - 4ru_{uu}u_{tr\phi}k^2 \\
& + 2r^2u_{\phi u}u_{trr}k^2 - 4Mr u_{\phi u}u_{trr}k^2 - 4r^2uu_{trr\phi}k^2 + 8Mr uu_{trr\phi}k^2 + 2r^2u_{t\phi}u_{tt}k^2 + 4Mr u_{t\phi}u_{tt}k^2 \\
& + 2r^2u_{tu}u_{tt\phi}k^2 + 4Mr u_{tu}u_{tt\phi}k^2 - r^2u_{\phi u}u_{ttt}k^2 - 2Mr u_{\phi u}u_{ttt}k^2 - r^2uu_{ttt\phi}k^2 - 2Mr uu_{ttt\phi}k^2 + \\
& 8Mr u_{t\phi}^2k - 4Mr u_{tu}u_{t\phi\phi}k + 2Mr u_{\phi u}u_{tt\phi}k + 2Mr uu_{tt\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi}u_t - \\
& 2(M-r)(k^2+r^2-2Mr)u_{r\phi}u_t + r^4u_{rr\phi}u_t - 4Mr^3u_{rr\phi}u_t + 4M^2r^2u_{rr\phi}u_t - 2r^2u_{\theta\theta}u_{t\phi} + \\
& 4Mr u_{\theta\theta}u_{t\phi} - 4r^3u_{ru}u_{t\phi} + 12Mr^2u_{ru}u_{t\phi} - 8M^2ru_{ru}u_{t\phi} - 2r^4u_{rr}u_{t\phi} + 8Mr^3u_{rr}u_{t\phi} - \\
& 8M^2r^2u_{rr}u_{t\phi} + r^2u_{\phi u}u_{t\theta\theta} - 2Mr u_{\phi u}u_{t\theta\theta} - 2r^2uu_{t\theta\theta\phi} + 4Mr uu_{t\theta\theta\phi} + 2r^3u_{\phi u}u_{tr} - 6Mr^2u_{\phi u}u_{tr} \\
& + 4M^2ru_{\phi u}u_{tr} - 4r^3uu_{tr\phi} + 12Mr^2uu_{tr\phi} - 8M^2ru_{tr\phi} + r^4u_{\phi u}u_{trr} - 4Mr^3u_{\phi u}u_{trr} + \\
& 4M^2r^2u_{\phi u}u_{trr} - 2r^4uu_{trr\phi} + 8Mr^3uu_{trr\phi} - 8M^2r^2uu_{trr\phi} + 2r^4u_{t\phi}u_{tt} + 2r^4u_{tu}u_{tt\phi} - r^4u_{\phi u}u_{ttt} \\
& - r^4uu_{ttt\phi}])
\end{aligned} \tag{3.26}$$

$$\begin{aligned}
\mathcal{Q}_6 &= u_{t\phi\phi} \\
\Phi_6^t &= -\frac{1}{6(k^2+r(r-2M))\sin(\theta)} [u_{\phi\phi}(2k^2\cos(\theta)^2 - (k^2+r(r-2M))\sin(\theta)(k^2\sin(\theta)u_{tt} - u_{\theta})\cos(\theta) + \\
& 2r(r-2M) + \sin(\theta)^2(u_{rr}k^4 + 2r^2u_{rr}k^2 - 4Mr u_{rr}k^2 - r^2u_{tt}k^2 - 2Mr u_{tt}k^2 - 4Mr u_{t\phi}k + \\
& (k^2+r(r-2M))u_{\theta\theta} - 2(M-r)(k^2+r^2-2Mr)u_r + r^4u_{rr} - 4Mr^3u_{rr} + 4M^2r^2u_{rr} - r^4u_{tt})) \\
& + \sin(\theta)(\sin(\theta)(-3r(r^3+k^2(2M+r))u_{tu}u_{t\phi\phi} - u_{\phi}(u_{rr\phi}k^4 + 2r^2u_{rr\phi}k^2 - 4Mr u_{rr\phi}k^2 - r^2u_{tt\phi}k^2 \\
& - 2Mr u_{tt\phi}k^2 + 2Mr u_{t\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi} - 2(M-r)(k^2+r^2-2Mr)u_{r\phi} + r^4u_{rr\phi} \\
& - 4Mr^3u_{rr\phi} + 4M^2r^2u_{rr\phi} - r^4u_{tt\phi}) + u(u_{rr\phi\phi}k^4 + 2r^2u_{rr\phi\phi}k^2 - 4Mr u_{rr\phi\phi}k^2 + 2r^2u_{tt\phi\phi}k^2 + \\
& 4Mr u_{tt\phi\phi}k^2 + 2Mr u_{t\phi\phi\phi}k + (k^2+r(r-2M))u_{\theta\theta\phi\phi} - 2(M-r)(k^2+r^2-2Mr)u_{r\phi\phi} + r^4u_{rr\phi\phi} \\
& - 4Mr^3u_{rr\phi\phi} + 4M^2r^2u_{rr\phi\phi} + 2r^4u_{tt\phi\phi})) + (k^2+r(r-2M))\cos(\theta)(-3\sin(\theta)u_{tu}u_{t\phi\phi}k^2 + \\
& u_{\phi}(k^2\sin(\theta)u_{tt\phi} - u_{\theta\phi}) + u(2\sin(\theta)u_{tt\phi\phi}k^2 + u_{\theta\phi\phi})))]
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
& +12Mr^2u_{ru}{}_{t\phi} - 8M^2ru_{ru}{}_{t\phi} - 2r^4u_{rru}{}_{t\phi} + 8Mr^3u_{rru}{}_{t\phi} - 8M^2r^2u_{rru}{}_{t\phi} + r^2u_{\phi}u_{t\theta\theta} - \\
& 2Mr_{\phi}u_{t\theta\theta} - 2r^2uu_{t\theta\theta\phi} + 4Mr_{uu}{}_{t\theta\theta\phi} + 2r^3u_{\phi}u_{tr} - 6Mr^2u_{\phi}u_{tr} + 4M^2ru_{\phi}u_{tr} - 4r^3uu_{tr\phi} + \\
& 12Mr^2uu_{tr\phi} - 8M^2ruu_{tr\phi} + r^4u_{\phi}u_{trr} - 4Mr^3u_{\phi}u_{trr} + 4M^2r^2u_{\phi}u_{trr} - 2r^4uu_{trr\phi} + \\
& 8Mr^3uu_{trr\phi} - 8M^2r^2uu_{trr\phi} + 2r^4u_{t\phi}u_{tt} - r^4u_{tu}{}_{t\phi} - r^4u_{\phi}u_{ttt} + 2r^4uu_{ttt\phi}] \\
& \hspace{15em} (3.28)
\end{aligned}$$

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